Econ 703 TA Note 5

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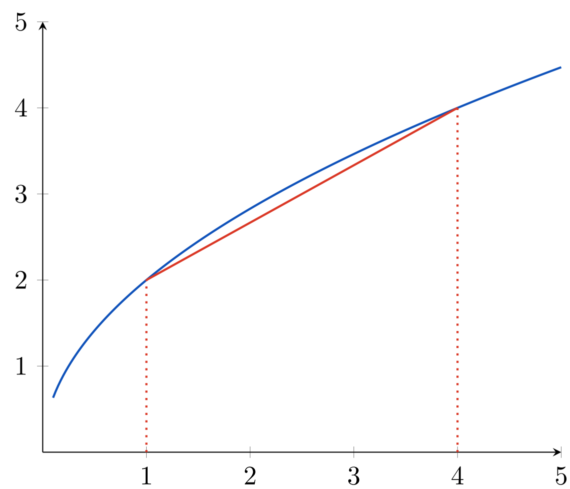
1 Concave and Convex Functions on $\mathbb{R}$

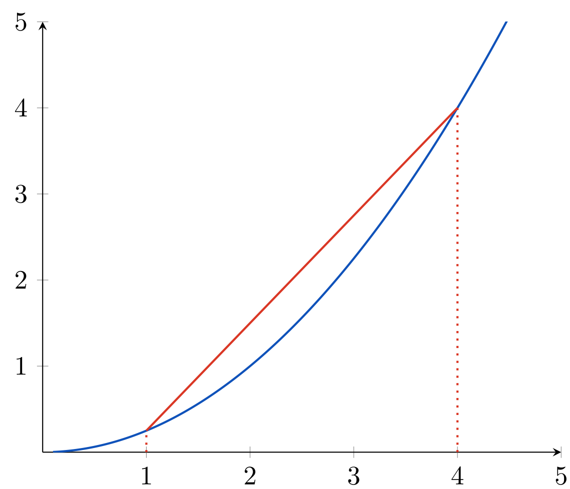
Definition 1.1 (Concave): A function $f : (a, b) \to \mathbb{R}$ is said to be concave (convex) if for all $x \neq y \in (a, b)$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$
$$(\leq)$$

If the inequality is strict for any $x \neq y$ and $\lambda \in (0, 1)$, then we say that f is strictly convex (concave). Below are graphs of a concave and a convex function.



(a) A Concave Function



(b) A Convex Function

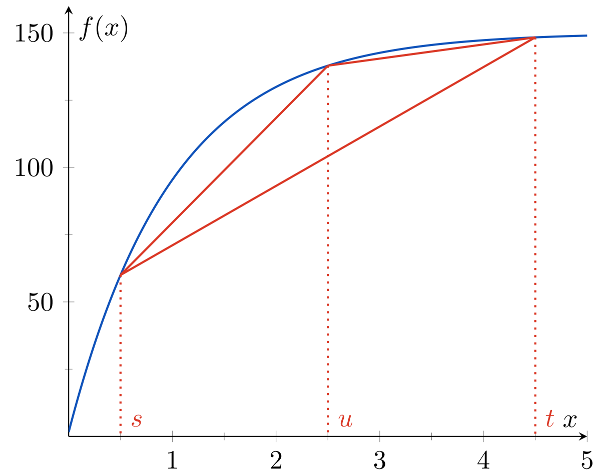
From now on, we state all theorems in terms of concave functions.

Theorem 1.1: Let $f : (a, b) \to \mathbb{R}$ be concave. Then for all $a < s < u < t < b$, we have

$$\frac{f(u) - f(s)}{u - s} \geq \frac{f(t) - f(s)}{t - s} \geq \frac{f(t) - f(u)}{t - u}$$

The inequalities are strict if f is strictly concave.

[*] This TA note was prepared for the Econ PhD math camp taught by Prof. John Kennan at UW-Madison in 2025. All errors are mine.



Proof. There exists $\lambda \in (0,1)$ such that $u = \lambda s + (1-\lambda)t$. Then by concavity of f,

$$f(\lambda s + (1-\lambda)t) \geq \lambda f(s) + (1-\lambda)f(t).$$

Observe that

$$\begin{aligned}
\frac{f(u)-f(s)}{u-s} &= \frac{f(\lambda s + (1-\lambda)t) - f(s)}{(\lambda-1)s + (1-\lambda)t} \\
&\geq \frac{(\lambda-1)f(s) + (1-\lambda)f(t)}{(\lambda-1)s + (1-\lambda)t} = \frac{f(t)-f(s)}{t-s}, \\
\frac{f(t)-f(u)}{t-u} &= \frac{f(t) - f(\lambda s + (1-\lambda)t)}{\lambda t - \lambda s} \\
&\leq \frac{\lambda f(t) - \lambda f(s)}{\lambda t - \lambda s} = \frac{f(t)-f(s)}{t-s},
\end{aligned}$$

where we used the previous inequality to get the two inequalities. □

The following graph illustrates the theorem.

2 Right and Left Derivatives and Subgradient

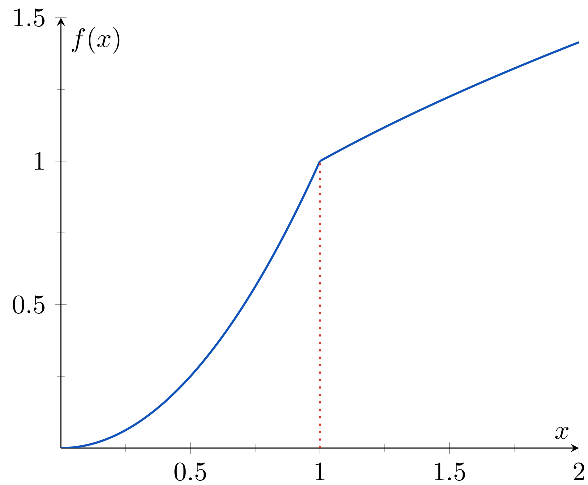
Definition 2.1 (Right and Left Derivative): A function $f : (a,b) \to \mathbb{R}$ is said to be **right (left) differentiable** at c if, for any sequence $\{x_n\}$ with $x_n > (<)c$ and $x_n \to c$, the limit

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c}$$

exists and is the same for all such sequence. This common value is called the **right (left) derivative** of f at c, denoted by $f'(c+)$ $(f'(c-))$.

Remark: A function may be both right differentiable and left differentiable at c, yet still fail to be differentiable at c if the right and left derivatives are not equal. The following graph illustrates a function that has both right and left derivatives at $x = 1$, but is not differentiable at $x = 1$.

Theorem 2.1: Let $f : (a,b) \to \mathbb{R}$ be concave. Then f is both right and left differentiable at any point $c \in (a,b)$. Moreover, $f'(c-) \geq f'(c+)$.



Proof. Fix $c \in (a, b)$. We prove that f is right differentiable. Consider the set of slopes:

$$A = \left\{ \frac{f(x) - f(c)}{x - c} : x \in (a, b),\ x > c \right\}.$$

This set is bounded from above by $\frac{f(c)-f(k)}{c-k}$ where $k = (c+a)/2$ by Theorem 1.1. Hence, it has a supremum: $v = \sup A$. We show that v is the right derivative. Let $x_n > c$ and $x_n \to c$, and let $\epsilon > 0$. By the definition of supremum, there exists $z > x$ such that

$$\frac{f(z) - f(c)}{z - c} > v - \epsilon.$$

Since $x_n \to c$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x < x_n < z$, and thus

$$v \geq \frac{f(x_n) - f(c)}{x_n - c} \geq \frac{f(z) - f(c)}{z - c} > v - \epsilon$$

by Theorem 1.1. We have thus proved

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = v.$$

One can show similarly that f is left differentiable with left derivative $\inf B$ where

$$B = \left\{ \frac{f(x) - f(c)}{x - c} : x \in (a, b),\ x < c \right\}.$$

Since for all $a \in A, b \in B$, $a \geq b$ by Theorem 1.1, we have $f'(c+) = \sup A \leq \inf B = f'(c-)$. ☐

Corollary (Continuity of a Concave Function): Let $f : (a, b) \to \mathbb{R}$ be concave. Then f is continuous at all inner points, namely, f is continuous on (a, b).

Proof. Fix $c \in (a, b)$. Since f has a right derivative, for any sequence $x_n > c$ with $x_n \to c$,

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = f'(c+).$$

Because $x_n - c \to 0$, it follows that $f(x_n) - f(c) \to 0$. Similarly, if $x_n < c$ and $x_n \to c$, then $f(x_n) - f(c) \to 0$ as well. Thus, for any sequence $x_n \to c$, we obtain

$$\lim_{n\to\infty} f(x_n) = f(c).$$

☐

Definition 2.2 (Subgradient): Let $f : (a, b) \to \mathbb{R}$ be concave. For any $c \in (a, b)$, a number $v \in [f'(c+),\ f'(c-)]$ is called a **subgradient** of f at c. The interval $[f'(c+),\ f'(c-)]$ is called the **subdifferential** of f at c.

Theorem 2.2: Let $f : (a, b) \to \mathbb{R}$ be concave, and let v be a subgradient of f at c. Then the tangent line

$$h(x) = f(c) + v(x - c)$$

lies above $f(x)$, i.e., $h(x) \geq f(x)$ for all $x \in (a, b)$.

Proof. For any $x > c$,

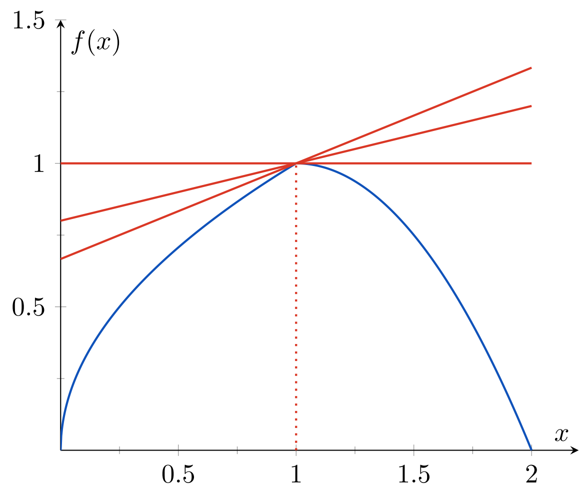
$$\frac{f(x) - f(c)}{x - c} \leq f'(c+) \leq v \implies f(x) \leq f(c) + v(x - c).$$

For any $x < c$,

$$\frac{f(x) - f(c)}{x - c} \geq f'(c-) \geq v \implies f(x) \leq f(c) + v(x - c).$$

□

The following graph illustrates Theorem 2.2. If f is concave, then every tangent line at a point lies above the graph of the function.



3 Extreme Points

Theorem 3.1 (Necessary and Sufficient Condition for Maximal Points): Let $f : (a, b) \to \mathbb{R}$ be a concave function. Then $c \in (a, b)$ is a global maximal point **if and only if** 0 is a subgradient of f. Namely, $0 \in [f'(c+),\ f'(c-)]$.

Proof. ($\implies$): Suppose $f'(c-) < 0$. Recall that

$$f'(c-) = \inf\left\{\frac{f(x) - f(c)}{x - c} : x < c\right\}.$$

Therefore, there exists $x < c$ such that $\frac{f(x)-f(c)}{x-c} < 0$. But then this implies $f(x) - f(c) > 0$, a contradiction. Hence, $f'(c-) \geq 0$. Similarly, one can prove that $f'(c+) \leq 0$.

($\Longleftarrow$): By Theorem 2.2, $h(x) = f(c) + 0(x - c) = f(c) \geq f(x)$ for all $x \in (a, b)$. □

Theorem 3.2: Let $f : (a, b) \to \mathbb{R}$ be a **strictly** concave function. Then f has at most one global maximal point.

Proof. Assume $x \neq y$ are both global maximal points, $f(x) = f(y) = c$. Consider $z = 0.5x + 0.5y$. Then $f(z) > 0.5f(x) + 0.5f(y) > c$, a contradiction. □

Theorem 3.3: Let $f : (a, b) \to \mathbb{R}$ be a concave function. If f has a **global minimal point**, then f must be constant.

Proof. We show that if $c \in (a, b)$ is a global minimal point, then $f(x)$ must be a constant function on (a, b). Take any $x, y \in (a, b)$ such that $x < c < y$. Since c lies strictly between a and b, we can write

$$c = \lambda x + (1 - \lambda)y$$

for some $\lambda \in (0, 1)$. By concavity,

$$f(c) = f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).$$

This implies $f(c) \geq f(x)$ or $f(c) \geq f(y)$. Without loss of generality, assume $f(c) \geq f(x)$.

Since c is a global minimal point, we also have $f(c) \leq f(x)$ and $f(c) \leq f(y)$. Hence, $f(c) = f(x)$. Substituting into the concavity inequality gives

$$f(c) \geq \lambda f(c) + (1 - \lambda)f(y) \implies f(c) \geq f(y).$$

Therefore, we also have $f(y) = c$. We conclude that $f(x) = f(y) = c$. □