Econ 703 Note 4

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August 20, 2025

1 Topology of a General Metric Space

Definition 1.1 (Open Ball): Let (X, d) be a metric space. For any point $x \in X$ and $\delta > 0$, the set

$$B(x, \delta) = \{y \in X : d(x, y) < \delta\}$$

is called an open ball centered at x with radius δ.

Definition 1.2 (Open and Closed Sets): A subset $O \subset X$ is *open* in X if for all $x \in O$, there exists $\delta > 0$ such that $B(x, \delta) \subset O$. A subset $C \subset X$ is *closed* in X if its complement $X \setminus C$ is open.

It is important to note that whether a set is open or closed depends on the metric space we are working in. The collection of all open sets in a metric space is called the **topology** of the metric space.

Example 1.1: Under the Euclidean metric, $(0, 1]$ is not open in $\mathbb{R}$, however, it is open in $[-1, 1]$.

Example 1.2: Recall the metric in TA Note 2:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Every subset of $\mathbb{R}$ is an open set under this metric.

Theorem 1.1 (Characterization of Closed Sets): Let (X, d) be a metric space and $A \subset X$. A is closed if and only if whenever a sequence $\{x_n\}$ in A converges to some $x \in X$, we have $x \in A$.

Theorem 1.2: Let (X, d) be a complete metric space. Then $A \subset X$ is closed if and only if (A, d) is a complete metric space.

2 Continuous Functions on a General Metric Space

Definition 2.1: Let $f : (X, d_X) \to (Y, d_Y)$. f is said to be continuous if for any sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x$, we have

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

*This TA note is based on Prof. John Kennan's math camp lecture in 2025 at UW-Madison. All errors are mine.

3 Continuous Functions on $\mathbb{R}$

3.1 Extreme Value Theorem

Theorem 3.1: Let $f : A \to \mathbb{R}$ be **continuous** where $A \subset \mathbb{R}$ is **closed and bounded**. Then $f(A)$ is closed and bounded.

Proof. It suffices to prove that $f(A)$ is closed and bounded.

First show that $f(A)$ is bounded. Suppose $f(A)$ is not bounded from above. Then there exists a sequence $\{y_n\} \subset f(A)$ such that

$$\lim_{n\to\infty} y_n = +\infty.$$

There exists a corresponding sequence $\{x_n\} \subset A$ such that $y_n = f(x_n)$. Since A is bounded, by Bolzano-Weierstrass Theorem, there exists a subsequence such that $\{x_{n_k}\}$ that converges to some point x. And since A is closed, $x \in A$. By continuity,

$$\lim_{k\to\infty} f(x_{n_k}) = f(x) < \infty.$$

But $\{f(x_{n_k})\}_{k=1}^{\infty}$ as a subsequence of $\{y_n\}$, a sequence that goes to infinity, should also go to infinity. A contradiction. We can also show $f(A)$ is bounded from below using the same method.

Next we show that $f(A)$ is closed. Let $\{y_n\} \subset f(A)$ and suppose that it converges to y. We want to show $f(x) = y$ for some $x \in A$. There exists a corresponding sequence $\{x_n\} \subset A$ such that $y_n = f(x_n)$. Since A is closed and bounded, again by Bolzano-Weierstrass Theorem, there exists a subsequence $\{x_{n_k}\}$ that converges to some point $x \in A$. By continuity,

$$\lim_{n\to\infty} f(x_{n_k}) = f(x).$$

And since $\{f(x_{n_k})\}$ is a subsequence of $\{y_n\}$, they must have the same limit, therefore $f(x) = y$. □

Corollary (Extreme Value Theorem): Let $f : A \to \mathbb{R}$ be **continuous** where $A \subset \mathbb{R}$ is **closed and bounded**. Then f admits a maximum and minimum in A. Namely, there exists $\overline{a} \in A$ such that $f(\overline{a}) \geq f(a)$ for all $a \in A$. There also exists $\underline{a} \in A$ such that $f(\underline{a}) \leq f(a)$ for all $a \in A$.

3.2 Intermediate Value Theorem

Theorem 3.2 (Intermediate Value Theorem): Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose $f(a) < f(b)$. Then for **any** $\xi \in (f(a), f(b))$, there exists $c \in (a, b)$ such that $f(c) = \xi$.

Proof. Consider the set $A = \{x \in [a, b] : f(x) \leq \xi\}$. First note that $a \in A$ and $b \in [a, b] \setminus A$. So A and $[a, b] \setminus A$ are both nonempty. Since A is nonempty and bounded, $c = \sup A$ exists. We claim that $f(c) = \xi$. Fix $\epsilon > 0$. There exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. By Proposition 2.2 in Note 3, there exists $x \in A$ such that $|x - c| < \delta$. Hence, we have

$$f(c) \leq f(x) + |f(x) - f(c)| \leq \xi + \epsilon.$$

On the other hand, since $A \setminus [a, b]$ is not empty and $\xi < b$, there exists $y \in A \setminus [a, b]$ such that $|y - c| < \delta$. Hence, we also have

$$f(c) \geq f(y) - |f(y) - f(c)| \geq \xi - \epsilon.$$

We conclude that

$$\xi - \epsilon \leq f(c) \leq \xi + \epsilon.$$

Since this holds for all $\epsilon > 0$, $f(c) = \xi$. □