Econ 703 TA Note 3

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August 15, 2025

1 Sets and Functions

1.1 Set

A set is a **non-ordered** collection of objects. We can define a set by naming all objects in the set. For example $S = \{1, 2, 3\}$. We can also define a set by describing it:

$$S = \{s : P(s)\},$$

where $P(\cdot)$ is a predicate. This set collects all objects s such that $P(s)$ is true. Here is a list of commonly used notation in set theory.

1. $x \in A$: x is an element of A, or equivalently, A includes x.
2. $A \subset B$: A is a subset of B, or equivalently, B contains A.
3. $A \cup B$: The union of A and B. x belongs to $A \cup B$ if and only if $x \in A$ **or** $x \in B$.
4. $A \cap B$: The intersection of A and B. x belongs to $A \cap B$ if and only if $x \in A$ **and** $x \in B$.
5. $\bigcup_{\lambda \in I} A_\lambda$: The union of all sets A_λ indexed by some $\lambda \in I$ where I is some index set (I can be finite, infinite, countable, or uncountable). x belongs to $\bigcup_{\lambda \in I} A_\lambda$ if and only if $x \in A_\lambda$ **for some** $\lambda \in I$.
6. $\bigcap_{\lambda \in I} A_\lambda$: The intersection of all sets A_λ indexed by some $\lambda \in I$. x belongs to $\bigcap_{\lambda \in I} A_\lambda$ if and only if $x \in A_\lambda$ **for all** $\lambda \in I$.
7. $A \setminus B$: This notation only appears when $B \subset A$. x belongs to $A \setminus B$ if $x \in A$ and $x \notin B$.
8. A^c: Suppose there is a universal set X. $A^c = X \setminus A$, the complement of A. x belongs to A^c if x is not in A.
9. $A \times B$: The Cartesian product of A and B. $A \times B = \{(a, b) : a \in A, b \in B\}$.
10. $\bigtimes_{\lambda \in I} A_\lambda$: The Cartesian product of A_λ's.

Theorem 1.1 (De Morgan's Law): The following is true

1. $(A \cup B)^c = A^c \cap B^c$.
2. $(A \cap B)^c = A^c \cup B^c$.
3. $\left(\bigcup_{\lambda \in I} A_\lambda\right)^c = \bigcap_{\lambda \in I} A_\lambda^c$.
4. $\left(\bigcap_{\lambda \in I} A_\lambda\right)^c = \bigcup_{\lambda \in I} A_\lambda^c$.

*This TA note is based on Prof. John Kennan's math camp lecture in 2025 at UW-Madison. All errors are mine.

1.2 Functions

A **function** is a rule that assigns to each element of a set A (called the **domain**) exactly one element of a set B (called the **codomain**). If f is a function from A to B, we write

$$f : A \to B.$$

And for all element $x \in A$, we write $f(x) \in B$ as the corresponding element in B.

Definition 1.1: We say that a function $f : A \to B$ is

- one-to-one (injective), if any two points in A are mapped to different points in B. For all $a, b \in A$, $a \neq b \implies f(a) \neq f(b)$.
- onto (surjective), if every point in B is mapped by some point in A. For all $b \in B$, there exists $a \in A$ such that $f(a) = b$
- bijective, if f is one-to-one and onto.

Let $f : A \to B$. For any subset of the domain, $A' \subset A$, the **image** of A is defined as,

$$f(A') = \{b \in B : b = f(a) \text{ for some } a \in A'\}.$$

It is the set of all points in B mapped by some point in A'. The image of the domain, $f(A)$, is called the **range** of the function. f is onto if and only if $f(A) = B$, that is, the range equals the codomain. For any subset of the codomain, $B' \subset B$, the **preimage** of B' is defined as,

$$f^{-1}(B') = \{a \in A : f(a) \in B'\}.$$

It is the set of all points in A being mapped into B'. It is clear that $f^{-1}(B) = A$.

1.3 Cardinality of a Set

The cardinality of a set refers to its size. For a finite set, this is simply the number of its elements. For infinite sets, we compare cardinalities using functions. The notation $|A|$ stands for the cardinality of a set A.

Definition 1.2: For any two sets A and B, we say that $|A| \geq |B|$ if there exists an **onto** function $f : A \to B$.

An equivalent definition is that there exists a **one-to-one** function $f : B \to A$.

Example 1.1: The cardinality of $[0, 1]$ is the same as the cardinality of $\mathbb{R}$.

Example 1.2: Surprisingly, $|\mathbb{R}^n| = |\mathbb{R}|$ for all $n \in \mathbb{N}$. There exists an onto function from $\mathbb{R}$ to $\mathbb{R}^n$!

Definition 1.3: The **power set** of a set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A. That is,

$$\mathcal{P}(A) = \{S : S \subseteq A\}.$$

Theorem 1.2 (Cantor): Let A be any set. Then the power set of A, $\mathcal{P}(A)$ has a strictly larger cardinality than A, $|\mathcal{P}(A)| > |A|$.

Proof. We want to show that there does not exist an onto function $f : A \to \mathcal{P}(A)$. Assume the contrary that there exists an onto function $f : A \to \mathcal{P}(A)$. Consider the set

$$E = \{a \in A : a \notin f(a)\}.$$

Since f is onto, there exists $e \in A$ such that $E = f(e)$.

1. If $e \in E$: Then by the definition of E, $e \notin f(e) = E$. A contradiction.
2. If $e \notin E$: Then $e \notin f(e)$. By the definition of E, $e \in E$. A contradiction.

Since we reached a contradiction in either cases, there cannot exist such f. □

Starting from $\mathbb{N}$, and iterating the power set,

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \ldots,$$

gives an infinite strictly increasing chain of distinct cardinalities by Theorem 1.2. Thus, there are infinite different “sizes of infinity”.

Definition 1.4 (Countable): We say that a set A is countable if $|A| \leq |\mathbb{N}|$. If a set is not countable, we say that it is uncountable.

Theorem 1.3: A **countable union** of countable sets remains countable. Any **finite Cartesian product** of countable sets is still countable.

Example 1.3: $\mathbb{Q}$ is countable. $\mathbb{R}$ is uncountable by Cantor's diagonal argument.

2 The Real line $\mathbb{R}$

2.1 Supremum, Infimum and The Axiom of Completeness

Definition 2.1 (Bounded): For any set $A \subset \mathbb{R}$, we say that A is **bounded from above** if there exists $b \in \mathbb{R}$ such that $b \geq a$ for all $a \in A$. Such b is called an **upper bound** of A. On the other hand, we say that A is **bounded from below** if there exists $c \in \mathbb{R}$ such that $c \leq a$ for all $a \in A$. Such c is called a **lower bound** of A. If a set is both bounded from above and bounded from below, we say that it is **bounded.**

A bounded set can have many upper bounds and many lower bounds. There is one upper (lower) bound of special interest: the **supremum** (**infimum**).

Definition 2.2 (Supremum): Suppose $A \subset \mathbb{R}$ is bounded from above. The **smallest** among all its upper bounds is called the **supremum** (or **least upper bound**), denoted $\sup A$.

Definition 2.3 (Infimum): Suppose $A \subset \mathbb{R}$ is bounded from below. The **largest** among all its lower bounds is called the **infimum** (or **greatest lower bound**), denoted $\inf A$.

How do we know that for a set that is bounded from above (below), the supremum (infimum) exists? In fact, we don't — we **assume** it. This is the **axiom of completeness,** a foundational property of the real numbers. An axiom cannot be proven; it is accepted as a starting point on which the rest of the theory is built.

Definition 2.4 (Axiom of Completeness): Every nonempty set $A \subset \mathbb{R}$ that is bounded from above has a supremum in $\mathbb{R}$. Similarly, every nonempty set $A \subset \mathbb{R}$ that is bounded from below has an infimum in $\mathbb{R}$.

Wait — we have used the word *complete* in two contexts: the **axiom of completeness** and the notion of **a complete metric space.** How are these two concepts related? Later we will see that the Axiom of Completeness actually implies that $\mathbb{R}$ is a complete metric space under the Euclidean metric. Recall that $\mathbb{Q}$ is *not* a complete metric space. Essentially, $\mathbb{R}$ is the "completion" of $\mathbb{Q}$ — the smallest complete set that contains $\mathbb{Q}$.

Here we state an important yet straightfoward result:

Proposition 2.1: For any set $A \subset \mathbb{R}$, if $\sup A$ exists, then it is unique. Same for $\inf B$.

2.2 Sequences in $\mathbb{R}$ and the Bolzano-Weierstrass Theorem

Definition 2.5 (Monotone): A sequence $\{x_n\} \subset \mathbb{R}$ is called monotone if it is increasing: $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, or if it is decreasing: $x_n \geq x_{n+1}$ for all $n \in N$.

Here is a powerful result that basically relies on pure logic (and does not rely on the Axiom of Completeness).

Theorem 2.1: Every sequence in $\mathbb{R}$ has a monotone subsequence.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. We say that x_n is a **peak** if any subsequent term is less than or equal to it, namely, $x_m \leq x_n$ for all $m \geq n$. So, if x_n is not a peak, it means that $x_m > x_n$ for some $m > n$. We separate two cases:

- The sequence has infinite peaks: Write $\{x_{n_k}\}_{k=1}^{\infty}$ be the sequence of peaks. By the definition of peaks,

  $$x_{n_1} \geq x_{n_2} \geq x_{n_3} ...$$

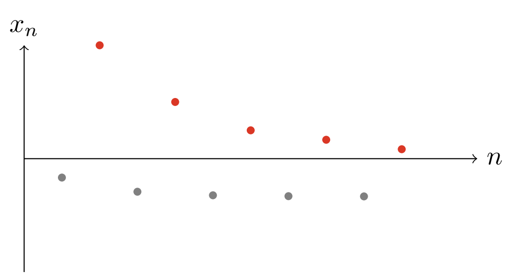
  Henceforth, the sequence of peaks is an increasing subsequence.

- The sequence has finite peaks: This means that **we can find the last peak** in the sequence. Suppose x_n is the last peak. We construct a decreasing sequence as follows: We know that x_{n+1} is not a peak. Therefore, $\exists m_1 > n+1$ such that $x_{n+1} < x_{m_1}$. x_{m_1} is also not a peak, and thus $\exists m_2 > m_1$ such that $x_{m_1} < x_{m_2}$. x_{m_2} is again not a peak, and thus $\exists m_3 > m_2$ such that $x_{m_2} < x_{m_3}$. Inductively, we construct a strictly increasing sequence

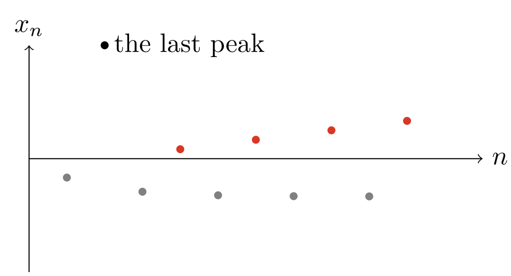
  $$x_{m_1} < x_{m_2} < x_{m_3} < ...$$

  $\{x_{m_k}\}_{k=1}^{\infty}$ is a monotone subsequence. (If the sequence has no peak at all, we can just start the process from x_1).

Since the statement is true for both cases, the proof is done. The figures below illustrate the construction of a monotone subsequence in each case, where the red dots indicate the selected elements for the subsequence. ☐



(a) Infinite Peaks



(b) Finite Peaks

The following is a useful characterization of the supremum.

Proposition 2.2: For any set $A \subset \mathbb{R}$, $a = \sup A$ if and only if

1. For all $\epsilon > 0$, there exists $x \in A$ such that $x > a - \epsilon$.
2. For all $\epsilon > 0$, $x < a + \epsilon$ for all $x \in A$.

Similarly, $a = \inf A$ if and only if

1. For all $\epsilon > 0$, there exists $x \in A$ such that $x < a + \epsilon$.
2. For all $\epsilon > 0$, $x > a - \epsilon$ for all $x \in A$.

The proof of the following theorem relies on the **Axiom of Completeness**.

Theorem 2.2 (Monotone Convergence Theorem): Any bounded monotone sequence in $\mathbb{R}$ converges. Specifically, any increasing sequence converges to its supremum, and any decreasing sequence converges to its infimum.

Proof. Let $\{x_k\}$ be a bounded increasing sequence in $\mathbb{R}$. By the Axiom of Completeness, $x = \sup\{x_k\}$ exists. Fix $\epsilon > 0$. By Proposition 2.2, there exists $x_n > x - \epsilon$. But since $\{x_k\}$ is increasing, $x_k > x - \epsilon$ for all $k \geq n$. Therefore, $x - \epsilon < x_k < x + \epsilon$ for all $k \geq n$.

The proof is identical for a bounded decreasing sequence. □

Theorem 2.3 (Bolzano-Weierstrass Theorem): Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. The conclusion follows immediately from combining Theorem 2.1 and Theorem 2.2. □

2.3 Completeness of $\mathbb{R}$

With Bolzano-Weierstrass Theorem established, we are ready to prove that $\mathbb{R}$ is a complete metric space. Note that Bolzano-Weierstrass Theorem holds because of the Axiom of Completeness: the proof of the theorem uses Monotone Convergence Theorem, whose proof, in turn, depends on the Axiom of Completeness.

Theorem 2.4: $\mathbb{R}$ is a complete metric space under the Euclidean metric.

Proof. Let $\{x_k\}_{k=1}^{n}$ be a Cauchy sequence in $\mathbb{R}$. We first show that it is bounded, and then we apply the Bolzano-Weierstrass Theorem. There exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $|x_n - x_m| < 1$. So, starting from x_N, all subsequent terms in the sequence is bounded from above by $x_N + 2$, and bounded from below by $x_N - 2$. Therefore, the whole sequence is bounded from above by $\max\{x_1, ..., x_{N-1}, x_N + 2\}$ and bounded from below by $\min\{x_1, ..., x_{N-1}, x_{N-2}\}$.

By Bolzano-Weierstrass Theorem (Theorem 2.3), $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Say it converges to x. Fix $\epsilon > 0$. There exists M_1 such that for all $n_k > M_1$, $|x_{n_k} - x| < \epsilon/2$. Since $\{x_n\}$ is Cauchy, there exists M_2 such that for all $n, m \geq M_2$, $|x_n - x_m| < \epsilon/2$. Let K be such that $n_K > \max\{M_1, M_2\}$. Then for all $n \geq M_1$,

$$|x_n - x| \leq |x_n - x_{n_K}| + |x_{n_K} - x| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

2.4 Limit Superior and Limit Inferior

Definition 2.6 (Limsup): Let $\{x_n\}$ be bounded from above and let $y_n = \sup_{k \geq n} x_k$. $\lim_{n\to\infty} y_n$ is called the limit sup of $\{x_n\}$, denoted $\limsup_{n\to\infty} x_n$.

Remark: Note that $\{y_n\}$ is a decreasing sequence. Therefore, the existence of $\lim_{n\to\infty} y_n$ is guaranteed by Monotone Convergence Theorem.

We can similarly define $\liminf_{n\to\infty} x_n$ for a sequence $\{x_n\}$ that is bounded from below.

Theorem 2.5: A sequence $\{x_n\}$ converges if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x.$$

Proof. We first prove the if part. Suppose

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x.$$

Let $\epsilon > 0$. Since $\limsup_{n\to\infty} x_n = x$, there exists $n_1 \in \mathbb{N}$ such that $x - \epsilon < \sup_{k\geq n} x_k < x + \epsilon$. Similarly, there exists $n_2 \in \mathbb{N}$ such that $x - \epsilon < \inf_{k\geq n} x_k < x + \epsilon$. Take $n^* = \max\{n_1, n_2\}$. Since $\sup_{k\geq n} x_k$ is decreasing in n, we have $\sup_{k\geq n^*} x_k < x + \epsilon$. Since $\inf_{k\geq n} x_k$ is increasing in n, we have $\inf_{k\geq n^*} x_k > x - \epsilon$. Therefore, for all $l \geq n^*$,

$$x - \epsilon < \inf_{k\geq n^*} x_k \leq x_l \leq \sup_{k\geq n^*} x_k < x + \epsilon.$$

□