Econ 703 TA Note 12

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1 Separating Hyperplane Theorem

Definition 1.1 (Hyperplane): Let $v \in \mathbb{R}^n$ and $r \in \mathbb{R}$. The hyperplane $\mathcal{H}(v, r) \subset \mathbb{R}^n$ is defined to be the set

$$\mathcal{H}(v, r) \coloneqq \{x \in \mathbb{R}^n : v \cdot x = r\}.$$

A hyperplane separates the whole space into two **closed half-spaces**:

$$\mathcal{H}_+(v, r) \coloneqq \{x \in \mathbb{R}^n : v \cdot x \geq r\}$$
$$\mathcal{H}_-(v, r) \coloneqq \{x \in \mathbb{R}^n : v \cdot x \leq r\}.$$

The intersection of the two closed half-spaces is exactly the hyperplane. The interior of the two closed half-spaces are called the **open half-spaces**:

$$\mathcal{H}^o_+(v, r) \coloneqq \{x \in \mathbb{R}^n : v \cdot x > r\}$$
$$\mathcal{H}^o_-(v, r) \coloneqq \{x \in \mathbb{R}^n : v \cdot x < r\}.$$

Definition 1.2: Let $\mathcal{H}(v, r)$ be a hyperplane in $\mathbb{R}^n$. Two subsets A and B are **weakly separated** by $\mathcal{H}(v, r)$ if A lies in one closed half-space and B lies in the other. Namely, either one of the two cases hold:

1. $A \subset \mathcal{H}_+(v, r)$ and $B \subset \mathcal{H}_-(v, r)$.
2. $A \subset \mathcal{H}_-(v, r)$ and $B \subset \mathcal{H}_+(v, r)$.

Definition 1.3: Let $\mathcal{H}(v, r)$ be a hyperplane in $\mathbb{R}^n$. Two subsets A and B are **strictly separated** by $\mathcal{H}(v, r)$ if A lies in one open half-space and B lies in the other. Namely, either one of the two cases hold:

1. $A \subset \mathcal{H}^o_+(v, r)$ and $B \subset \mathcal{H}^o_-(v, r)$.
2. $A \subset \mathcal{H}^o_-(v, r)$ and $B \subset \mathcal{H}^o_+(v, r)$.

Theorem 1.1 (Separating Hyperplane Theorem): Let A and B be two **closed disjoint convex** sets in $\mathbb{R}^n$. Then there exists a hyperplane $\mathcal{H}(v, r)$ that **weakly separates** A and B. Moreover, if at least one of A or B is **compact**, then there exists a hyperplane that **strictly separates** the two.

*This TA note is based on Prof. John Kennan's math camp lecture taught in 2025 at UW-Madison. All errors are mine.

2 Blackwell Sufficient Condition

Let $X \subset \mathbb{R}^n$. We will let $B(X)$ denote the set of all bounded functions defined on X. For any bounded function f on X, the number

$$\|f\| \coloneqq \sup\{|f(x)| : x \in X\}$$

exists. Note that for any f and g in $B(X)$, $f - g$ is still in $B(X)$. Therefore,

$$\|f - g\| = \sup\{|f(x) - g(x)| : x \in X\}$$

is well-defined. One can check that $d(f, g) \coloneqq \|f - g\|$ is indeed a metric on $B(X)$. This makes $(B(X), \|\cdot\|)$ a metric space. We will write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Theorem 2.1 (Blackwell's Sufficient Condition for a Contraction): Let $T : (B(X), \|\cdot\|) \to (B(X), \|\cdot\|)$ be a mapping satisfying

(i) (Monotonicity): $f, g \in B(X)$ and $f \leq$ implies $T(f) \leq T(g)$.

(ii) (Discounting): There exists some $\beta \in (0, 1)$ such that

$$[T(f + a)] \leq T(f) + \beta a$$

for all $f \in B(X)$, $a \geq 0$.

Then, T is a contraction mapping on $B(K)$ with modulus β.

Proof. Let $f, g \in B(X)$, and write $a = \|f - g\|$. Note that $f + a \geq g$. By monotonicity, $T(f + a) \geq T(g)$. By discounting, $T(f) + \beta a \geq T(f + a) \geq T(g)$. We then have $T(g) - T(f) \leq \beta a$. Similarly, $T(f) - T(g) \leq \beta a$. Combining the two, we have $\|T(f) - T(g)\| \leq \beta a = \beta \|f - g\|$. $\square$