Econ 703 Note 11

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1 Linear Independence and Basis

Definition 1.1 (Linear Independence): Let V be a vector space and $S = \{v_1, ..., v_n\} \subset V$. We say that S is **linearly dependent** if there exists $\{\alpha_i\} \subset \mathbb{R}$ not all 0 such that

$$\sum_{i=1}^{n} \alpha_i v_i = 0.$$

If S is not linearly dependent, we say that it is **linearly independent**.

Definition 1.2 (Spanning): Let V be a vector space and $S = \{v_1, ..., v_n\} \subset V$. We say that S spans V if for all $v \in V$, there exists $\{\alpha_i\} \subset \mathbb{R}$ such that

$$v = \sum_{i=1}^{n} \alpha_i v_i.$$

Example 1.1: In $V = \mathbb{R}^2$, $\{(1,0),(1,1)\}$ is linearly independent. On the other hand, $\{(1,1),(2,2)\}$ is linearly dependent. $\{(1,0),(1,1)\}$ spans $\mathbb{R}^2$, while $\{(1,1),(2,2)\}$ doesn't.

Definition 1.3 (Basis): Let V be a vector space. $\mathcal{B} = \{b_1, ..., b_n\}$ is called a **basis** of V if

(i) $\mathcal{B}$ is linearly independent.

(ii) $\mathcal{B}$ spans V.

Example 1.2: $\{(1,1),(1,0)\}$ is a basis of $\mathbb{R}^2$.

Theorem 1.1: Let $\mathcal{B}$ be a basis of V. Any $v \in V$ can be written as a **unique** linear combination of vectors in $\mathcal{B}$.

Definition 1.4 (Vector Representation): Let V be a vector space and $\mathcal{B}$ be a basis of it. For any

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$v \in V$, say $v = \sum_{i=1}^{n} \alpha_i b_i$, define the vector representation of v to be the $n \times 1$ real vector

$$[v]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \end{bmatrix}^{\intercal}.$$

Theorem 1.2: Suppose a finite set $\mathcal{B}$ is a basis of V. Then

1. Any basis of V has the same number of elements as $\mathcal{B}$. Such number is called the **dimension** of V, denoted by $\dim(V)$.
2. Any linearly independent set of vectors S with $|S| = \dim(V)$ is a basis of V.
3. Any set of vectors S with $|S| > \dim(V)$ is linearly dependent.
4. Any set of vectors S that spans V satisfies $|S| \geq \dim(V)$.

2 Linear Transformation and Matrix Representation

Definition 2.1 (Linear Transformation): Let V and W be vector spaces. We say that a function $T : V \to W$ is a **linear transformation** if

1. For all $v_1, v_2 \in V$, $T(v_1 + v_2) = T(v_1) + T(v_2)$.
2. For all $\alpha \in \mathbb{R}, v \in V$, $T(\alpha v) = \alpha T(v)$.

Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis of V. If $T : V \to W$ is a linear transformation, then T is fully characterized by $T(b_1), ..., T(b_n)$: For any $v \in V$, since $\mathcal{B}$ is a basis, there exists α_i's such that $v = \sum_{i=1}^{n} \alpha_i v_i$. We then have

$$T(v) = \sum_{i=1}^{n} \alpha_i T(b_i).$$

Definition 2.2 (Matrix Representation): Let $T : V \to V$ be a linear transformation and let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis of V. Suppose for all j,

$$T(b_j) = \sum_{i=1}^{n} \alpha_{ij} b_i.$$

The matrix representation of T under $\mathcal{B}$, denote by $[T]_{\mathcal{B}}$ is the n by n matrix with its (i, j) element being a_{ij}.

Example 2.1: Let $\mathcal{B} = \{(1, 0), (0, 1)\}$ and $\mathcal{B}' = \{(1, 0), (1, 1)\}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = (x_1 + 2x_2, 2x_1 + 2x_2)$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad [T]_{\mathcal{B}'} = \begin{bmatrix} -1 & -1 \\ 2 & 4 \end{bmatrix}.$$

Theorem 2.1: Let V be a vector space and $\mathcal{B} = \{b_1, ..., b_n\}$ a basis. Let $T : V \to V$ be a linear transformation. For any $v \in V$,

$$[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}}.$$

A particularly useful basis for $\mathbb{R}^n$ is the standard basis: $\mathcal{E} = \{e_1, e_2, ..., e_n\}$. For any vector $v = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n$, $[v]_{\mathcal{E}} = v$. By the above theorem,

$$T(v) = [T(v)]_{\mathcal{E}} = [T]_{\mathcal{E}}[v]_{\mathcal{E}} = [T]_{\mathcal{E}}v.$$

Therefore, any linear transformation T on $\mathbb{R}^n$ is essentially a matrix.

3 Eigenvalues, Eigenvectors and Characteristic Polynomial

Since any linear transformation T is essentially a matrix, our further discussion on eigenvalues and eigenvectors will be based on matrices.

Definition 3.1: Let $A \in \mathbb{R}^{n\times n}$. We say that $\lambda \in \mathbb{R}$ is an **eigenvalue** of A if there exists $v \in \mathbb{R}^n$ and $v \neq 0$ such that

$$Av = \lambda v.$$

It is equivalent to saying that there exists $v \neq 0$ such that $(A - \lambda I_n)v = 0$. Any v that satisfies $Av = \lambda v$ is called an **eigenvector** of A corresponding to the eigenvalue λ.

Recall the following result:

Theorem 3.1: For $M \in \mathbb{R}^{n\times n}$, the following are equivalent:

1. M is invertible, i.e., M^{-1} exists.
2. $Mv = 0 \implies v = 0$.
3. M's column vectors form a basis of $\mathbb{R}^n$.
4. M's row vectors form a basis of $\mathbb{R}^n$.
5. $\det(M) \neq 0$.

This gives us the following characterization of an eigenvalue of a square matrix A.

Theorem 3.2: Let $A \in \mathbb{R}^{n\times n}$. $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Definition 3.2 (Characteristic Polynomial): The characteristic polynomial of $A \in \mathbb{R}^{n\times n}$ is defined as $p_A(x) = \det(A - xI)$.

Therefore, λ is an eigenvalue of A if and only if it is a root of $p_A(x)$.

4 Diagonalization

Definition 4.1: A square matrix $A \in \mathbb{R}^{n\times n}$ is said to be **diagonalizable** if there exists an invertible P and a diagonal D such that

$$A = PDP^{-1}.$$

Let P be a $n \times n$ matrix with its column vectors $\{v_1, ..., v_n\}$ being the eigenvectors of $A \in \mathbb{R}^{n\times n}$. Then

$$\begin{aligned} AP = A\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} &= \begin{bmatrix} Av_1 & \cdots & Av_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix} \\ &= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= PD. \end{aligned}$$

If additionally, P is invertible, then $A = PDP^{-1}$. Together with Theorem 3.1, this gives us the following result:

Theorem 4.1: A square matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable if and only if there exists a set of eigenvectors of A, $\{v_1, ..., v_n\}$, that is a basis of $\mathbb{R}^n$.

5 Symmetric Matrices

Theorem 5.1 (Spectral Theorem): Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then A is diagonalizable by an orthonormal matrix. That is, there exists P and a diagonal matrix D such that

$$A = PDP^\intercal, \quad P^\intercal = P^{-1}$$

where the diagonal entries of D are the eigenvalues of A.

Definition 5.1 (Positive Definite): A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called

- positive (semi-)definite if $v^\intercal Av > (\geq) 0$ for all $v \in \mathbb{R}^2, v \neq 0$.
- negative (semi-)definite if $v^\intercal Av < (\leq) 0$ for all $v \in \mathbb{R}^2, v \neq 0$.
- indefinite if $v^\intercal Av < 0, u^\intercal Au > 0$ for some $u, v \in \mathbb{R}^2$.

Theorem 5.2: A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is

- positive (semi-)definite if all of its eigenvalues are $> (\geq) 0$.
- negative (semi-)definite if all of its eigenvalues are $< (\leq) 0$.
- indefinite if some eigenvalues < 0 while some eigenvalues > 0.

Proof. We prove the case when A is positive definite. By the Spectral Theorem, we can write $A = PDP^\intercal$, where D is a diagonal matrix and the diagonal entries are eigenvalues of A, $\lambda_1, ..., \lambda_n$. (The eigenvalues may repeat.) For any $v \in V, v \neq 0$,

$$v^\intercal Av = v^\intercal PDP^\intercal v = (P^\intercal v)^\intercal D(P^\intercal v).$$

Write $u = P^\intercal v$. Then

$$v^\intercal Av = u^\intercal Du = \sum_{i=1}^{n} \lambda_i u_i^2.$$

Suppose all λ_i's are positive. For all $v \neq 0$, since $P^\intercal$ is invertible, $u = P^\intercal v = (u_1, ..., u_n)$ is nonzero. Therefore,

$$v^\intercal Av = \sum_{i=1}^{n} \lambda_i u_i^2 > 0.$$

Suppose A is positive definite. For $u = (0, 0, ..., \underbrace{1}_{i^{th}}, 0, ..., 0) \in \mathbb{R}^n$, since $P^\intercal$ is invertible, there exists $v \neq 0$ such that $P^\intercal v = u$. Therefore,

$$\lambda_i = \sum_{i=1}^{n} \lambda_i u_i^2 = v^\intercal Pv > 0.$$

Since the argument holds for all i, $\lambda_i > 0$ for all $1 \leq i \leq n$. $\square$