Econ 703 Note 10

Chia-Min Wei [*]

October 23, 2025

1 Correspondence

Definition 1.1: A **correspondence** $\mathcal{D} : \Theta \to \mathcal{P}(S)$ maps any point $x \in \Theta$ to a subset of S. The **graph** of the correspondence is defined to be the set

$$\mathrm{Gr}(\mathcal{D}) = \{(x, s) \in \Theta \times S : s \in \mathcal{D}(x)\}.$$

A correspondence $\mathcal{D}$ is said to be **compact-valued** if $\mathcal{D}(x)$ is compact (i.e., closed and bounded) for all $x \in \Theta$.

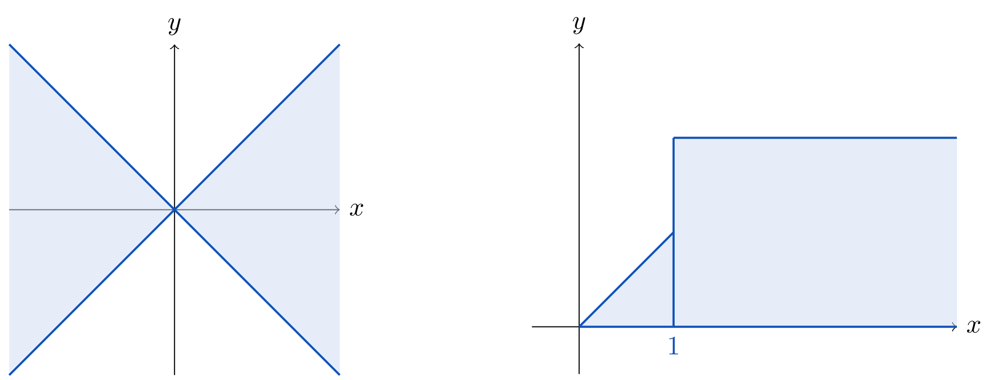
Example 1.1: Define $\mathcal{D} : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{D}(x) = [-x, x].$$

Figure 1a shows the graph of $\mathcal{D}(x)$. Define $\tilde{\mathcal{D}} : [0, \infty) \to \mathcal{P}(\mathbb{R})$ by

$$\tilde{\mathcal{D}}(x) = \begin{cases} [0, x] & \text{if } x < 1 \\ [0, 2] & \text{if } x \geq 1. \end{cases}$$

Figure 1b shows the graph of $\tilde{\mathcal{D}}(x)$. Both correspondences are compact-valued.



(a) $\mathcal{D}(x) = [-x, x]$ (b) Graph of $\tilde{\mathcal{D}}(x)$

Figure 1: Examples of Correspondence

[*]This TA note is based on Prof. John Kennan's math camp lecture taught in 2025 at UW-Madison. All errors are mine.

Example 1.2: Let $\Theta = \{(p_1, ..., p_n) \in \mathbb{R}^n : p_i > 0 \ \forall i\}$ and $S = \{(x_1, ..., x_n) : x_i \geq 0 \ \forall i\}$. Fix $I > 0$. The correspondence $\mathcal{D} : \Theta \to \mathcal{P}(S)$ defined by

$$\mathcal{D}(p) = \{(x_1, ..., x_n) \in \mathbb{R}^n : p_1 x_1 + ... + p_n a_n \leq I, x_i \geq 0 \ \forall i\}$$

for all $p = (p_1, ..., p_n) \in \Theta$ is a compact-valued correspondence.

2 Continuity of a Correspondence

Now we define the notion of continuity when a correspondence is compact-valued.

Definition 2.1 (Upper-Semicontinuity): Let $\mathcal{D} : \Theta \to \mathcal{P}(S)$ be compact-valued. We say that $\mathcal{D}$ is upper-semicontinuous at $c \in \Theta$ if for all $x_n \to c$, $a_n \in \mathcal{D}(x_n)$,

(i) $\{a_n\}$ is bounded.

(ii) If $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n \in \mathcal{D}(c)$.

$\mathcal{D}$ is upper-semicontinuous at c means that, any smooth path (x_n, a_n) in the image of $\mathcal{D}$ with its first coordinate approaching c converges to a point (c, a) with $a \in \mathcal{D}(c)$.

Definition 2.2 (Lower-Semicontinuity): Let $\mathcal{D} : \Theta \to \mathcal{P}(S)$ be compact-valued. We say that $\mathcal{D}$ is lower-semicontinuous at $c \in \Theta$ if for all $x_n \to c$ and $a \in \mathcal{D}(c)$, there exists $a_n \in \mathcal{D}(x_n)$ such that $a_n \to a$.

$\mathcal{D}$ is lower-semicontinuous at c means that, for any point (c, a) with $a \in \mathcal{D}(c)$ and any sequence $x_n \to c$, there is a path (x_n, a_n) that completely lies in the graph of $\mathcal{D}$ and converges to (c, a).

Example 2.1: Define $\mathcal{D} : [0, \infty) \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{D}(x) = \begin{cases} [0, x] & \text{if } 0 \leq x < 1, \\ [0, 2] & \text{if } 1 \leq x < 2, \\ [0, 1] & \text{if } x \geq 2. \end{cases}$$

$\mathcal{D}$ is usc but not lsc at $x = 1$. On the other hand, $\mathcal{D}$ is lsc but not usc at $x = 2$. Figure 2 shows the graph of $\mathcal{D}$. We cannot find a path in the graph of $\mathcal{D}$ approaching $(1, 1.6)$ from the left. On the other hand, the path shown in the graph has its first coordinate approaching 2, but it does not converge to a point $(2, a)$ with $a \in \mathcal{D}(2)$.

Definition 2.3: If a correspondence is $\mathcal{D} : \Theta \to \mathcal{P}(S)$ both usc and lsc at a point $c \in \Theta$, then we say that it is continuous at c.

Definition 2.4 (Single-valued): We say that a correspondence $\mathcal{D} : \Theta \to \mathcal{P}(S)$ is **single-valued** if $\mathcal{D}(x)$ is a singleton for all $x \in \Theta$.

A single-valued correspondence $\mathcal{D} : \Theta \to \mathcal{P}(S)$ is basically a function from Θ to S.

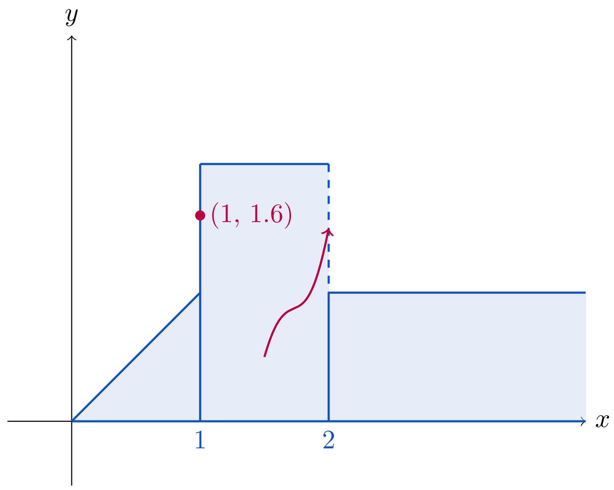


Figure 2: A Correspondence

Theorem 2.1: A single-valued correspondence that is semicontinuous (whether usc or lsc) is continuous when viewed as a function. Conversely, every continuous function, when viewed as a single-valued correspondence, is both usc and lsc.

3 The Maximum Theorem

Theorem 3.1 (The Maximum Theorem): Let $f : S \times \Theta \to \mathbb{R}$ be a continuous function and $\mathcal{D} : \Theta \to \mathcal{P}(S)$ be a compact-valued, continuous correspondence. Define for all $x \in \Theta$,

$$V(x) = \max\{f(s,x) : s \in \mathcal{D}(x)\},$$
$$\mathcal{D}^*(x) = \{s \in \mathcal{D}(x) : f(s,x) = V(x)\}.$$

Then V is a continuous function on Θ, and $\mathcal{D}^*(x)$ a compact-valued, usc correspondence on Θ.

Remark: When $\mathcal{D}^*$ is single-valued, then under the assumptions of the theorem, $\mathcal{D}^*$ can be viewed as a continuous function.

The theorem says that if the objective function is continuous and the choice set changes continuously as the parameter changes, the value of the maximization problem also changes continuously. Furthermore, if the maximizer is unique for each parameter, the optimal point varies continuously with the parameter.